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D-dimensional arrays of Josephson junctions, spin glasses and q-deformed harmonic oscillators

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Abstract. We study the statistical mechanics of a D-dimensional array of Josephson junctions in the presence of a magnetic field. In the high-temperature region, the thermodynamical properties can be computed in the limit $D \to \infty$, where the problem is simplified; this limit is taken in the framework of the mean-field approximation. Close to the transition point, the system behaves very similarly to a particular form of spin glasses, i.e. to gauge glasses. We have noticed that in this limit, the evaluation of the coefficients of the high-temperature expansion may be mapped onto the computation of some matrix elements for the q-deformed harmonic oscillator.

1. Introduction

In this paper we are interested in studying the statistical mechanics of arrays of Josephson junctions in D-dimensions in the limit where $D \to \infty$. We will construct here the solution of the mean-field theory in the high-temperature phase. We postpone to a later stage the computation of the corrections to the mean-field approximation and the study of the low-temperature phase. The model has been studied in two dimensions, especially in the low-temperature region [1, 2], but no results are known for very high dimensions.

The model we consider is described by the Hamiltonian

$$H = -c(D) \sum_{j,k} \overline{\phi_j} U_{j,k} \phi_k + \text{HC}. \tag{1}$$

Here $c(D)$ is a normalization constant, which will be useful later to rescale the Hamiltonian in order to obtain a non-trivial limit when D goes to infinity. The spins ϕ_j are defined on a D-dimensional hypercubic lattice.

We can consider three possibilities:

(i) the spins ϕ_j are constrained to be of modulus one;

(ii) the spins ϕ_j have modulus one in the average at $\beta = 0$ (in this limit they have a Gaussian distribution); and

(iii) the spins satisfy the constraint $\sum_i |\phi_i|^2 = N$. This is the spherical model which is intermediate between the two previous models.

In the limit where the dimension D goes to infinity, the properties of the first and third model can be obtained from that of the Gaussian model. We will concentrate our attention on the Gaussian case.

The couplings U are non-zero only for nearest-neighbour sites. They are complex numbers of modulus one and they satisfy the relation

$$U_{k,j} = \overline{U_{j,k}}. \tag{2}$$

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In other words, the couplings U are the links variables of a $U(1)$ lattice gauge field.

We will select the couplings U to give a constant magnetic field. Many different orientations of the magnetic field can be chosen. For simplicity, we restrict our computation to the case where the flux through each elementary plaquette is given by B (or $-B$), independently of the plane to which the plaquette belongs. This corresponds to constant uniform frustration on all the plaquettes. In the extreme case ($B = \pi$), we obtain a fully-frustrated model, while for $B = 0$ we recover the ferromagnetic case. Random-point-dependent B values correspond to a particular form of spin glasses, i.e. to gauge glasses [3–6].

More precisely, we set

$$B_{\alpha,\beta} = S_{\alpha,\beta} B \tag{3}$$

where $S_{\alpha,\beta}$ may take the values 1 or -1 and $B_{\alpha,\beta}$ is the antisymmetric tensor corresponding to the magnetic field, which in the continuum limit is given by $\partial_\alpha A_\beta - \partial_\beta A_\alpha$. The ordered product of the four links of a plaquette in the α, β plane is equal to $\exp(\mathrm{i}B_{\alpha,\beta})$.

We must now specify $S_{\alpha,\beta}$, i.e. the sign of $B_{\alpha,\beta}$. A possible choice would be to take

$$S_{\alpha,\beta} = 1 \qquad \text{for} \qquad \alpha > \beta \tag{4}$$

which implies $B_{\alpha,\beta} = B$ for $\alpha > \beta$.

In two and three dimensions, this choice is equivalent to any other possible choice of the sign. In three dimensions, the magnetic field is a vector and all the vectors corresponding to different choices of the sign may be obtained from one another by rotation. The choice of S does not influence the thermodynamics.

In more than three dimensions, different choices of the matrix S are not equivalent† and we must select one among all the possible choices. In this paper, we consider the case in which the matrix S is generic, i.e. the signs of B are randomly chosen. The system is translationally invariant and the randomness appears in only the relative orientation of the magnetic field with the crystal axis.

In the two-dimensional case, we recover the usual description for an XY system (or equivalently an array of Josephson junctions) in a constant magnetic field.

The aim of this paper is to compute the statistical properties of this model in the mean-field approximation in the high-temperature region. The first difficulty we face consists of finding the spectral properties of the lattice-discretized Laplacian in the presence of a magnetic field. The lattice Laplacian is defined as

$$(\Delta f)_j = \sum_k U_{j,k} f_k. \tag{5}$$

The spectral properties of the lattice Laplacian in two dimensions have been carefully studied. They depend on the arithmetic properties of the B/π, i.e. different results are obtained for rational and irrational B/π [2].

The study of the lattice Laplacian in higher dimensions is much less developed. In any dimension, the explicit construction of the field U shows that for rational B/π, of the form $B = 2\pi r/s$ with both r and s integers, there is a gauge in which the U couplings are periodic functions of the position, with period s. In this case, the spectrum of the Laplacian has the typical band form, the edges of the bands being related to the eigenvalues of an

† I am grateful to E Marinari and F Ritort for crucial discussions on this problem.

$s^D \times s^D$ matrix. When both s and D are large, a direct study of the eigenvalues is rather complex.

We will study this problem in the limit of an infinite number of dimensions. We cannot solve it in a completely satisfactory way, but we can put forward some educated guesses. We will find some unexpected relations with the properties of the q-deformed harmonic oscillator. Finally, the behaviour of the model ends up being very similar to that of spin glasses. The reader should notice that it is not clear how much of our results survive in large, but finite, dimensions and that the properties of the model in high dimensions may be quite different from that of the two-dimensional model.

In section 2, we present some general considerations. In the next section, we show some general properties of the high-temperature expansion in the limit $D \to \infty$. We consider in detail the ferromagnetic case, the spin-glass case and the constant-frustration model. In section 4, we show the relation between the high-temperature expansion for the constant-frustration model in infinite dimension and the q-deformed harmonic oscillator. In the next section, we study the behaviour of our model near the critical point and we find that it is very similar to that of spin glasses. In section 6, we briefly discuss the problems related to the exchange of limits ($\beta \to \beta_c$ and $D \to \infty$). Finally (in the last section), we present our conclusions and express our points of view on the open problems. In the appendix we will describe some interesting features of the q-deformed harmonic oscillator, which shows an anomalous behaviour for $q = \exp(2\pi i\theta)$ when θ is rational.

2. General considerations

There are two extreme cases for the U which are very well studied for the Hamiltonian (1).

(i) We set

$$U_{j,k} = 1. \tag{6}$$

In this way, we obtain the usual ferromagnetic XY model. There is a ferromagnetic transition at $\beta = 1$ in the limit $D \to \infty$ if we set $c(D) = \frac{1}{2D}$, i.e. $c(D)$ has to be equal to the inverse of the coordination number of the hypercubic lattice.

(ii) We set

$$U_{j,k} = \exp(\mathrm{i}r_{j,k}) \tag{7}$$

where r are random numbers belonging to the interval 0–2π such that symmetry condition (2) is satisfied.

In this way, we obtain a spin-glass model of XY type, which is called a gauge glass [3–5]. The transition temperature is $\beta = 1$ in the limit $D \to \infty$ if we set $c(D) = (2D)^{-1/2}$, i.e. $c(D)$ is equal to the inverse of the square root of the coordination number.

The model we study is intermediate among the previous two problems. In order to define it properly, it is convenient to introduce the so-called Wilson loop. Let us consider a closed oriented circuit (C) on the lattice, which goes from the point j to the same point j and let us define $W(C)$ as the product of the U's along the circuit. The Wilson loop $W(C)$ is a gauge invariant. The knowledge of $W(C)$ for any C gives all gauge-invariant information concerning the gauge field.

In the continuum limit, we have

$$W(C) = \exp\left(\mathrm{i}\int_C \mathrm{d}x^\mu A_\mu(x)\right) = \exp(\mathrm{i}\Phi(C)) \tag{8}$$

where $\Phi(C)$ is the magnetic flux entangled within C.

In two dimensions, in the presence of a constant magnetic field, the Wilson loop is given by

$$W(C) = \exp(\mathrm{i}BS(C)) \tag{9}$$

where $S(C)$ is the signed area of the loop C.

In D dimensions, there are $D(D-1)/2$ planes oriented in the direction of the lattice. The choice of the magnetic field we study here is

$$W(C) = \exp(\mathrm{i}\Phi(C)) \qquad \Phi(C) = \sum_{\nu,\mu=\nu<\mu} S_{\nu,\mu}(C)B_{\mu\nu} \tag{10}$$

where the indices ν and μ denote one of the D possible different directions and $S_{\nu,\mu}$ is the signed area of the projection of curve C on the ν, μ plane.

As a consequence of gauge invariance, there are infinitely many choices of U which correspond to these Wilson loops. All these choices are physically equivalent. In two dimensions, we could set

$$U_1(j) = 1 \qquad U_2(j) = \exp(\mathrm{i}Bj_1)$$

where j_ν is the νth component of the vector j and we have introduced the short-hand notation

$$U_\nu(j) = U(j, j + n_\nu) \tag{11}$$

where n_ν is the unit vector in the ν direction.

This construction can be generalized to the D-dimensional case. For example, in four dimensions, one obtains

$$\begin{aligned} &U_1(j) = 1 \qquad U_2(j) = \exp(\mathrm{i}B_{21}j_1) \\ &U_3(j) = \exp(\mathrm{i}B_{31}j_1 + B_{32}j_2) \qquad U_4(j) = \exp(\mathrm{i}B_{41}j_1 + B_{42}j_2 + B_{43}j_3). \end{aligned} \tag{12}$$

Our main task will be the study of the associated Gaussian model, where the Hamiltonian is given by

$$H = -c(D)\sum_{j,k} \overline{\phi_j} U_{j,k}\phi_k + \mathrm{HC} - \tfrac{1}{2}\sum_k |\phi_k|^2. \tag{13}$$

The solution of this associated Gaussian model is a crucial step in the computation of the properties of the high-temperature expansion.

3. The high-temperature expansion

In the case of the Gaussian model, the free-energy density can be written as

$$\beta F(\beta) = \sum_C W(C)(\beta c(D))^{L(C)}/L(C) \tag{14}$$

where the sum is performed over all the closed lattice circuits with given starting point; $L(C)$ is the length of the circuit [6].

In a model (like the present one) where gauge-invariant quantities are translationally invariant [7], we can choose the origin (and the end) of the circuit at an arbitrary lattice point. In other cases, like spin glasses, we must average over all the possible starting points [8].

The previous formula can also be written as

$$\beta F(\beta) = \mathrm{tr}\ln(1 + c(D)\beta\Delta) = \sum_n \frac{(\beta c(D))^n}{n} \mathcal{N}(n)\langle W(C)\rangle_n \tag{15}$$

where we denote the average over all the circuits of length n by $\langle W(C)\rangle_n$ and the number of (rooted) closed circuits by $\mathcal{N}(n)$.

Differentiating the previous formulae, we obtain a similar result for the internal energy density:

$$2\beta c(D)U(\beta) = \sum_n (\beta c(D))^n \mathcal{N}(n)\langle W(C)\rangle_n. \tag{16}$$

Here the factor $1/n$ has disappeared.

3.1. The ferromagnetic case

This is the simplest case. We have only to compute $\mathcal{N}(n)$ since $\langle W(C)\rangle_n = 1$.

It is evident that $\mathcal{N}(n) = 0$ for odd n. The first non-zero contributions for small n are

$$\mathcal{N}(2) = 2D \qquad \mathcal{N}(4) = 6D(2D-1). \tag{17}$$

We could also compute $\mathcal{N}(n)$ using the representation

$$\mathcal{N}(n) = \int_B \mathrm{d}^D p \left(\sum_{\mu=1,D} 2\cos(p_\mu) \right)^n . \tag{18}$$

If we use the correct normalization of $c(D)$ that gives the critical temperature at 1, we immediately find that when $D \to \infty$ all these contributions vanish. This is a well known fact: in the high-temperature phase in the mean-field approximation, the internal energy of a ferromagnetic system is zero. The fluctuations contribute only in the subdominant terms of the large D expansion.

This behaviour implies that one should be careful in taking the limit $D \to \infty$. Indeed, it is easy to check that in the limit where $n \gg D$ one finds that [9]

$$c(D)^n \mathcal{N}(n) \propto n^{-D/2} \tag{19}$$

but, in the opposite limit $D \gg n$, one gets

$$N(n) \sim (n-1)!!(2D)^{n/2} \tag{20}$$

and, therefore,

$$c(D)^n \mathcal{N}(n) \sim \frac{(n-1)!!}{(2D)^{n/2}}. \tag{21}$$

Equation (20) is very simple to understand. In a closed circuit, for each step in one direction, there must be a step in the opposite direction. In infinite dimensions, all the steps are taken in different directions (in a way compatible with this constraint). The generic circuit will be thus identified by the directions in which these steps are performed (we have to make a choice $n/2$ times between these directions) and by the locations of the steps at which two opposite directions are chosen. In high dimensions, all the steps are performed in different directions and in this way one obtains the previous formula, i.e. the number of pairing of n objects ($(n-1)!!$) multiplied by the number of choices for the directions ($(2D)^{n/2}$).

If we were unaware of the correct normalization factor and we had put $c(D) = (\frac{1}{2D})^{1/2}$ with the aim of obtaining a non-trivial perturbative expansion, we would get the formula

$$\beta U(\beta) = \sum_n (2n-1)!!(\beta)^{2n}. \tag{22}$$

We would have found, in this way, that the high-temperature expansion has a zero radius of convergence. This is not a surprise [10] because in this scale the critical temperature is at $\beta = 0$ and any non-zero value of β is already in the low-temperature regime.

In the ferromagnetic case, the singularity of the free energy disappears when $D \to \infty$ in the high-temperature expansion with the correct $C(D)$. This effect can easily be explained. The ferromagnetic transition is characterized by the building up of a singularity at momentum $k = 0$ in the two-point correlation function. The free energy in the high-temperature phase is given by

$$f(\beta) \propto \int_B \mathrm{d}^D k \ln\left(1 - \beta \sum_{\nu=1,D} \cos(k_\nu)/(2D)\right) \tag{23}$$

where the integral is performed over the first Brillouin zone.

When $D \to \infty$, the region of momenta near the origin has a vanishing weight and its contribution to the singularity disappears. We can see a transition in the specific heat in the limit of infinite dimensions only if the directions of the most relevant modes are not orthogonal to the boundary of the Brillouin zone, where the measure is concentrated in momentum space.

3.2. *Spin glasses*

In this case, we will compute the spectrum of the random Laplacian. This can be done in the infinite-dimensional limit since we recover the old problem of computing the spectrum of a random matrix, which is given by a semicircular law†. Instead of directly using this result, we prefer to follow a diagrammatic approach.

In this case, the U's have zero average and are random elements of the $U(1)$ group. After the average over all the possible starting points, $W(C)$ gets contributions only from those circuits for which for any step going from i to k there is a step going from k to i. In other words, we must sum only over *backtracking* circuits.

Let us count the number of these circuits in infinite dimensions. We must compute

$$G_{2n} = \lim_{D\to\infty} (2D)^{-n} \mathcal{N}(2n) \langle W \rangle_{2n}. \tag{24}$$

† One could also use the replica approach.

It is easy to check that, for $n = 1$, we do not get any new contributions with respect to the previous case and $G_1 = 1$.

For larger values of n, a more detailed computation must be performed. To this end it is convenient to denote one of the different $2D$ possible directions in which a step could be performed by $a, b, c, \ldots$.

In the case $n = 2$, we have $3!!$ circuits which differ for the ordering possibilities:

$$aabb \qquad abba \qquad abab \tag{25}$$

where it is implicit that the second identical letter denotes a back step in the opposite direction of the first identical letter. We do not attach any meaning to the letters a or b: we could have written $aabb$ or $bbaa$ indifferently. In both cases, the second and fourth steps are in the opposite direction of the first and third step, respectively. (We neglect subleading terms for large D.)

Each of the $3!!$ choices correspond to $(2D)^2$ lattice circuits. The first two are backtracking circuits; the second is not. We, thus, find $G_2 = 2$.

In the case $n = 3$, we have $5!!$ circuits which differs for the ordering possibilities. We list here all the backtracking ones:

$$aabbcc \qquad abbcca \qquad abccba \qquad aabccb \qquad abccab. \tag{26}$$

Therefore, $G_3 = 5$. It is easy to verify that a circuit is backtracking if and only if the corresponding word may be reduced to the null word by subsequent elimination of consecutive identical letters.

The computation of G_n can, thus, be cast under the following graphical form. For each given word, we put its $2n$ letters (two by two equal), on a circle starting from a given point, in the same order as the letters of the corresponding word. We connect those points which have identical letters by a line and we count the number of intersections of the lines. This number is topologically invariant and does not depend on the point where the letter has been put on the circle, only on its order.

We can associate the number of intersections to each word. Let us call $I_n(m)$ the number of words which have m intersections ($m \leqslant n(n-1)/2$). It is easy to check that

$$I_n(0) = G_n. \tag{27}$$

Indeed, only in the case in which the resulting diagram is planar may the diagram be reduced to zero by consecutively removing equal letters.

The combinatorial problem of computing $I_n(0)$ has been solved [11] in the past†. After a short computation, one finds

$$I_n(0) = 4^n \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+2)} \tag{28}$$

The result of the computation can also be written in a slightly different form. We consider a Hilbert space and a base $(|m\rangle$ on this Hilbert space, where m ranges in the interval $[0\text{–}\infty]$). We define two shift operators $\mathcal{R}$ and $\mathcal{L}$ on this space

$$\mathcal{R}|m\rangle = |m+1\rangle \qquad \mathcal{L}|m\rangle = |m-1\rangle \tag{29}$$

† The result is a by-product of the formula relating the generating functionals of the connected and disconnected functions.

where $|-1\rangle$ is identified with the null vector.

These two operators satisfy the relation

$$\mathcal{L}\mathcal{R} = 1 \tag{30}$$

which is a particular case (for $q = 0$) of the q-deformed commutation relations†

$$\mathcal{L}\mathcal{R} - q\mathcal{R}\mathcal{L} = 1. \tag{31}$$

It is easy to see that

$$G_n = \langle 0|(\mathcal{R} + \mathcal{L})^{2n}|0\rangle \tag{32}$$

where the state $|0\rangle$ could also be characterized by the condition

$$\mathcal{L}|0\rangle = 0. \tag{33}$$

The existence of these two other formulations should not be a surprise. The condition of zero intersection implies that the diagram is planar and the theory of random matrices may be reformulated in terms of planar diagrams. The theory of random matrices can also be formulated in terms of the orthogonal polynomials with respect to a given measure [14] and in this context it is well known that the shift operators play a crucial role [15].

We finally find that

$$1 + \beta U(\beta) = \sum_n (4\beta^2)^n \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+2)} = \frac{1}{\pi}\int_{-2}^{2} d\lambda \, \frac{(1-\lambda^2/4)^{1/2}}{(1+\beta\lambda)}. \tag{34}$$

There is a transition at $\beta = \frac{1}{2}$, which is characterized by a singularity of the specific heat of the form $(\beta_c - \beta)^{-1/2}$. In other words, the critical exponent α is equal to $\frac{1}{2}$.

Equation (34) gives the result for spin glasses in the Gaussian approximation. Starting from equation (34), one can obtain the more familiar results for the Ising spin glass or for the spherical spin glass.

3.3. Josephson junctions in a magnetic field

In this case, we need first to compute the function

$$G_n(B) = \lim_{D\to\infty} (2D)^{-n} \mathcal{N}(2n) \langle W\rangle_n. \tag{35}$$

We will follow the strategy of first dividing the circuits into classes corresponding to different words of $2n$ letters (as in the previous case) and evaluate the contribution of each class.

Let us start by computing $G_2(B)$ (it is trivial that $G_1(B) = 1$). The backtracking circuits which correspond to the *planar* diagrams (the corresponding words are *aabb* and *abba*) give a contribution of 1 each. More generally, we can define the area of a circuit as the minimal area of a surface of lattice plaquettes which have that circuit as a boundary. Backtracking circuits can be characterized as area-zero circuits.

† In the case $q = 1$, we have bosonic commutation relations, for $q = -1$ we have fermionic commutation relations and for $q = \exp(i\theta)$ we have anionic commutation relations. Some applications of the anionic commutation relations can be found in [12, 13] and references therein.

For large D the word *abab* corresponds to $(2D)^2$ circuits with area 1. For half of these, the signed area (defined in equation (10)) $S(C)$ is equal to 1, for the other half $S(C)$ is equal to -1. If we recall that $W(C) = \exp(\mathrm{i}\Phi(C))$, the contribution of these circuits average to $\cos(B)$. We finally find

$$G_4(B) = 2 + q \tag{36}$$

where

$$q = \cos(B). \tag{37}$$

Generally speaking, each different word of length $2n$ is associated to $(2D)^n$ circuits having the same area. The signed area of these circuits having the same area (A) is different. In a large number of dimensions (in the generic case where all the independent steps are performed in different directions), the projected signed areas $S_{\mu,\nu}$ take only the values 0 or ± 1 and

$$\sum |S_{\mu,\nu}| = A. \tag{38}$$

If we average over all the possible orientations of the lattice, the contribution coming from circuits having the same word, we find that the average value of $\langle W(C) \rangle$ depends only on A and is given by

$$\langle W(C) \rangle_A = \left(\frac{\exp(\mathrm{i}B) + \exp(-\mathrm{i}B)}{2} \right)^A = q^A. \tag{39}$$

We finally find that

$$G_n(B) = \sum_w q^{A(w)} \tag{40}$$

where the sum is taken over all words of $2n$ letters and $A(w)$ is the area associated with each of these words.

We now show that the area of the circuit is exactly equal to the number of intersections of the lines connecting equal letters in the corresponding diagram. We can decrease the area by unity by interchanging two letters. For example

$$A(acdefbacdefb) = A(acdefabcdefb) + 1. \tag{41}$$

Indeed, the area of the projection on the a–b plane goes from 1 to 0 and the projected area on the other planes is the same in the two circuits corresponding to the two words. The same braiding operation decreases the number of intersections by 1. By subsequent operations of the previous kind, we can arrive at the zero-intersections case (planar diagrams) by decreasing both the area and the projection by unity each time. We have already remarked on the relation between the number of planar diagrams and the coefficient of the high-temperature expansion for spin glasses ($G_n = G_n(0)$).

We have, thus, transformed the problem of computing the high-temperature expansion into a combinatorial problem, which although not very easy, generalizes the computation of planar diagrams. The solution of this problem will be presented in the next section.

4. The q-deformed harmonic oscillator plays a role

We have reduced the problem of evaluating the high-temperature expansion for the Gaussian model in the presence of a magnetic field to the computation of the number of words of $2n$ letters, two by two equal, such that the number of intersections in the corresponding diagram is equal to a given number.

We claim that

$$G_n(B) = \sum_w q^{A(w)} = \langle 0|X^{2n}|0\rangle \tag{42}$$

where

$$X = \mathcal{R}_q + \mathcal{L}_q \tag{43}$$

and the operators $\mathcal{L}$ and $\mathcal{R}$ satisfy the commutation relations of a q-deformed harmonic oscillator

$$\mathcal{L}_q\mathcal{R}_q - q\mathcal{R}_q\mathcal{L}_q = 1. \tag{44}$$

Therefore, $\mathcal{L}_q$ may be identified with the destruction operator and R_q with the creation operator for a q-deformed harmonic oscillator. For $q = 1$, we recover the ferromagnetic case, for $q = -1$ the fully-frustrated case and for $q = 0$ the spin-glass case.

These operators may be represented as

$$\begin{aligned} \mathcal{R}_q|m\rangle &= [m]_q^{1/2}|m+1\rangle \\ \mathcal{L}_q|m\rangle &= [m-1]_q^{1/2}|m-1\rangle \end{aligned} \tag{45}$$

where

$$[m]_q = (1-q^{m+1})/(1-q) \tag{46}$$

and m ranges in the interval $[0\text{–}\infty]$. In the limit $q \to 1$, we obtain the usual bosonic oscillator and recover the usual formulae.

It is a simple matter of computation to verify that equation (4.2) gives

$$\begin{aligned} &G_1(B) = 1 \qquad G_1(B) = 2+q \qquad G_3(B) = 5+6q+3q^2+q^3 \\ &G_4(B) = 14+28q+28q^2+20q^3+10q^4+4q^5+q^6. \end{aligned} \tag{47}$$

These results coincide with the output of an explicit enumeration of the diagrams.

We have not been able to find a neat proof of equation (42). However, we have checked its validity in many special cases (large q, small q, $q = 1$, $q = 0$ and $q = -1$) and we are convinced of its validity.

Intuitively, equation (4.2) tells us that when we use the Wick theorem for q-deformed harmonic oscillators, we must bring together the different terms we contract, and for each term we obtain a factor q to the power of the number of objects we have to cross.

If we use this result, we finally find the quite simple formula

$$1 + \beta U(\beta) = \langle 0|\frac{1}{1+\beta X}|0\rangle_q \tag{48}$$

which gives a remarkable connection between the high-temperature behaviour of the Gaussian model and the q-deformed harmonic oscillator.

In this way, we have reduced the combinatorial problem of computing the high-temperature expansion to an algebraic problem.

5. Near the critical transition

The problem now is reduced to the computation of the spectrum of the operator X of the q-deformed harmonic oscillator. The computation is apparently non-trivial. We are, however, interested in the computation of the spectral density near the largest eigenvalues.

A simple case is $q = 1$, where the operator X_q is not bounded and the high-temperature expansion is divergent. In this case, X has a continuum spectrum and the highest eigenvalues of X are concentrated in the large m region. Let us assume that this feature is valid for q inside the interval $[-1, 1]$. One finds that

$$\mathcal{L}_q \sim (1-q)^{-1/2}\mathcal{L} \qquad \mathcal{R} \sim (1-q)^{-1/2}\mathcal{R}. \tag{49}$$

when the operator is applied to a state $|m\rangle$ in the region of large m. ($\mathcal{L}$ and $\mathcal{R}$ are the two shift operators for $q = 0$ which are used in the planar case.)

The difference among $\mathcal{L}_q$ and $(1-q)^{-1/2}\mathcal{L}$ can be seen only when the two operators act on a low-m state. It is reasonable to assume that the spectral radii and densities near the maximum eigenvalues are the same in the two cases. We have verified numerically that this conjecture is consistent (at least for q not too close to 1) by estimating the spectral density of X_q in subspaces of various size ($m < M$, with M up to 300).

We find, therefore, that the critical temperature is given by

$$\beta_c = \frac{(1-q)^{1/2}}{2} \tag{50}$$

which is the inverse of the spectral value of X, i.e.

$$|X|^2 = \frac{4}{(1-q)}. \tag{51}$$

The behaviour of the spectral density near the edge is the same as for the random matrix model, i.e. in spin glass. In this way, we find the same critical exponents as in spin glasses in the Gaussian approximation.

A possible physical interpretation is the following. In computing the internal energy, one has to sum over all the closed circuits. Circuits with large physical area average to zero and only fattened backtracking circuits survive. The situation is very similar to spin glasses, where only backtracking circuits contribute, the only effect being a renormalization of the temperature†.

6. The issue of exchanging limits

A very serious problem in assessing the relevance of these results is related to the exchange of the limits $D \to \infty$ and $\beta \to \beta_c$. If we exchange the limits, we become blind to any singularity whose strength vanishes in the limit $D \to \infty$. Sometimes this exchange is quite justified; sometimes it leads to disaster [16, 17].

The cases $q = 1$ and $q = -1$ are particularly instructive. The case $q = 1$ has already been discussed. The case $q = -1$ is quite interesting. We notice the following facts.

† The previous results imply that when n and m both approach infinity at fixed ratio, one finds $I_n(m) = I_n(0)\frac{(n+m)!}{n!m!}f(m/n)$. It is quite possible that this simple result has a direct proof.

(i) The spectrum of the lattice Laplacian for the fully-frustrated model is well known [17]. *A simple way to compute it consists of using the relation between the Gaussian fully-*frustrated model and the naive Wilson fermions on the lattice [18]. Indeed, let us start from the Hamiltonian of the naive Wilson fermions

$$H = \sum_i \left(\sum_\mu (\beta(\overline{\psi}(i+\hat{\mu}) - \overline{\psi}(i-\hat{\mu}))\gamma_\mu \psi(i)) + \overline{\psi}(i)\psi(i) \right) \tag{52}$$

where $\hat{\mu}$ is the versor in the μ direction, γ_μ are the appropriate Dirac gamma matrices in D dimensions (which satisfies the usual algebra) and ψ are the spinors on which these matrices act. For even D, the gamma matrices may be taken to have dimension $2^D/2$. In order to simplify the notation, we have not indicated the spinorial indices. If we introduce the field

$$\phi(i) = \prod_{\mu=1,D} \gamma_\mu^{i_\mu} \psi(i) \tag{53}$$

it is a well known fact that the lattice Dirac operator reduces to the Laplacian of a fully-frustrated model.

(ii) The previous remark implies that for $q = -1$, one has in the Gaussian approximation (with the appropriate rescaling of β)

$$1 + \beta U(\beta) = \int_B \mathrm{d}^D k \frac{1}{\left(1 - 2\beta^2 \sum_{\nu=1,D} \sin^2(k_\nu)/D\right)} \tag{54}$$

for all even values of the dimensions.

(iii) If we send D to infinity, we find that

$$1 + \beta U(\beta) = \frac{1}{1-\beta^2} \tag{55}$$

in perfect agreement with the direct computation. (In this case, the creation and annihilation operators act on a two-dimensional fermionic space.)

(iv) In any finite dimensions [17], the closest singularity to the origin of the function $U(\beta)$ is located at $\beta^2 = \frac{1}{2}$, which corresponds to the integration point where all the momenta are at the boundary of the Brillouin zone (i.e. $(k_\mu) = \pm\pi/2$).

(v) In infinite dimensions, the function $\beta_c(q)$ is discontinuous at $q = -1$. Indeed,

$$\lim_{q\to -1} \beta_c^2(q) = \tfrac{1}{2} \neq \beta_c^2(-1) = 1. \tag{56}$$

(vi) As already found in [17], at $q = -1$, the limit $D \to \infty$ of β_c is smaller by a factor of two than the value of β_c obtained from the high-temperature expansion computed directly at $D = \infty$. However, this difficulty seems to be confined to $q = -1$. If we first take the limit $D \to \infty$ at $q \neq 1$, we recover the correct critical point for the $q = 1$ case.

In other words, if we first compute the critical temperature at $D = \infty$ for $q \neq -1$, we obtain the correct value of the critical temperature at $q = -1$, while we would get the wrong results if we performed the limit $D \to \infty$ directly at $q = -1$. By consistency, we find that the prefactor in front of the nearest discontinuity vanishes when $q \to -1$, so that, for $q = -1$, this singularity disappears.

It seems that we are free to conjecture that (apart from two well understood problems at $q = -1$ [17] and $q = 1$ [10]) the correct value of the critical temperature is obtained when we send D to infinity first. A numerical verification of the validity of this conjecture may be attempted for $q = 0$ or $\pm\frac{1}{2}$, where β_c can be computed by diagonalizing matrices of size 2^D or 3^D, respectively.

7. Open problems

Let us suppose that the difficulties discussed in the previous section are not serious. We still face the problem of presenting a full computation of the high-temperature expansion in the XY model. We must include high-order terms which come from the fact that the distribution of the spins is not Gaussian. In the case of spin glasses, these corrections are relevant; however, they are identical in the Ising, XY and spherical model. In this last case, they can be computed by tuning the coefficient of the quadratic term in such way that the spherical constraint is satisfied.

We have not checked whether this also happens in our case, but it seems plausible. If this argument is correct, knowledge of the Gaussian propagator is sufficient to reconstruct the high-temperature expansion.

What happens in a finite number of dimensions is not clear. The first step is to verify whether the equality of the two-model survives in perturbation theory. Also if this check is satisfied, one should be very careful because of non-perturbative effects. It seems to me rather likely that for rational B the critical theory should behave differently from spin glasses and the only hope for having a spin-glass-like behaviour is for generic irrational B; however, I do not have solid arguments in this direction. It would be very interesting to connect this approach with the results obtained in two dimensions, where quantum groups have been used to compute the spectrum [19].

The possibility of having spin-glass behaviour for this non-random system [20] is fascinating and deserves more careful investigation.

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Appendix

In this short appendix, I report on some numerical findings that I have obtained on the behaviour of the spectral radius of X as function of θ for $q = \exp(\mathrm{i}2\pi\theta)$. In this case, I find a function which is discontinuous at all rational points, but the discontinuity vanishes when the rational point becomes irrational.

If we apply the previous formulae, we find that the spectral radius of X^2 should be

$$\frac{4}{|1-q|} = \left(\frac{4}{\sin^2(\pi\theta)}\right)^{1/2}. \tag{57}$$

The argument breaks down for rational θ. Indeed, if $\theta = r/s$, with both r and s integer (r and s are the smallest integer which have this property), X reduces to a finite-dimensional operator of size s. In this case, the previous formula is not correct. However, in the limit where s goes to infinity, it seems to become correct again. This can be seen by considering the function $R(\theta)$, defined as

$$|X|(\theta)^4 = \frac{4}{\sin^2(\pi\theta)}\left(1 - \frac{\pi^2}{2s^2}\right) + R(\theta). \tag{58}$$

The function $R(\theta)$ is the difference between the analytic continuation of the value of the spectral radius from $|q| < 1$ and the actual spectral radius (apart from the presence of a multiplicative factor which goes to zero as s^{-2} when $s \to \infty$ at fixed θ).

I have computed the function $R(\theta)$ for all rational with $s \leqslant 21$ (70 cases) and have found that it goes rapidly to zero with s (quite likely as s^{-2}). It seems likely that the function $R(\theta)$ is discontinuous at rational points, but the value of the discontinuity goes to zero when the rational becomes irrational (i.e. when $s \to \infty$).

Unfortunately, I am not aware of a physically interesting model in which the properties of X for complex q enter. This appendix should be considered as a curiosity.

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